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Determinants of Partition Matrices

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Let $\alpha_1, ..., \alpha_k$ be partitions of $2n$ with at least n 1's and $\beta_1, ..., \beta_k$ be partitions of $2n$ with exactly n parts. By M_n we denote the matrix whose entries m_{ij} are the number of ways to refine β_j into α_i. It is shown that det $M_n = 1$ for all n. © 1996

1. INTRODUCTION

Let $p(n)$ denote the partition function, i.e., the number of distinct ways to write n as a sum of positive integers. Partitions of n will be denoted by tuples of positive integers arranged in non-increasing order. For reasons that will become apparent later, we do not allow 1's in the tuples. The 1's of a partition can be determined by n which is considered as known, e.g. the partition of $n = 10$: $10 = 3 + 3 + 2 + 1 + 1$ will be written as (3, 3, 2).

Partitions are ordered by the lexicographical order, i.e., $(\alpha_1, ..., \alpha_m)$ is of higher rank than $(\beta_1, ..., \beta_k)$ if the first nonvanishing integer $\alpha_1 - \beta_1$, $\alpha_2 - \beta_2$, ... is positive.

Tuples can also be considered as set partitions in the following way. Let $\alpha = (\alpha_1, ..., \alpha_m)$ be a partition of n and $R = \{1, 2, ..., n\}$. α can be regarded as a set partition of $R = R_1 \cup \cdots \cup R_p$, where $R_1 = \{1, 2, ..., \alpha_1\}$, $R_2 = \{\alpha_1 + 1, \ \alpha_1 + 2, ..., \alpha_1 + \alpha_2\}$, ... (If the partition contains 1's, i.e., $n > \alpha_1 + \cdots + \alpha_m$, then $R_{m+1}, ..., R_p$ are singletons).

Now let $\alpha = (\alpha_1, ..., \alpha_m)$ and $\beta = (\beta_1, ..., \beta_k)$ be partitions of the same integer n. We say α is a refinement of β if the sets of β can be split up into smaller sets whose cardinalities are determined by the α_i. More precisely:

DEFINITION 1.1. $\alpha = (\alpha_1, ..., \alpha_m)$ is called a refinement of $\beta = (\beta_1, ..., \beta_k)$ if $\exists \phi$: $\{\alpha_1, ..., \alpha_m\} \to \{\beta_1, ..., \beta_k\}$ s.t.

$$\sum_{\alpha_i \in \phi^{-1}(\beta_j)} \alpha_i \leqslant \beta_j, \qquad \text{for} \quad 1 \leqslant j \leqslant k.$$

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By "the number of ways to refine β into α, ignoring order", we mean the number of ways to split up the sets of β into sets whose cardinalities are determined by the α_i. The order in which the sets are arranged is ignored.

EXAMPLE 1.2. Let $(4, 2)$ and $(2, 2)$ be partitions of 8. The number of ways to refine $(4, 2)$ into $(2, 2)$, ignoring order, is 9 since $\{1, 2, 3, 4\}$, $\{5, 6\}$, $\{7\}$, $\{8\}$ can be split up into two sets with two elements and singletons in 9 different ways:

$$\begin{aligned}
&\{1,2\},\{3,4\},\{5\},\{6\},\{7\},\{8\}; \quad \{1,3\},\{2,4\},\{5\},\{6\},\{7\},\{8\}\\
&\{1,4\},\{2,3\},\{5\},\{6\},\{7\},\{8\}; \quad \{1,2\},\{3\},\{4\},\{5,6\},\{7\},\{8\}\\
&\{1,3\},\{3\},\{4\},\{5,6\},\{7\},\{8\}; \quad \{1,4\},\{3\},\{4\},\{5,6\},\{7\},\{8\}\\
&\{2,3\},\{3\},\{4\},\{5,6\},\{7\},\{8\}; \quad \{2,4\},\{3\},\{4\},\{5,6\},\{7\},\{8\}\\
&\{3,4\},\{3\},\{4\},\{5,6\},\{7\},\{8\}
\end{aligned}$$

Notice that if $(4, 2)$ and $(2, 2)$ are considered as partitions of any integer $\geqslant 6$ the number of ways to refine will still be 9. It is true in general that the number of ways to refine one partition into another, ignoring order, does not depend on the number of 1's in the partition.

We are particularly interested in two special kinds of partitions. Let n be a fixed positive integer. A partition of $2n$ into exactly n parts is called a type I partition and a partition of $2n$ having at least n 1's is called a type II partition.

PROPOSITION 1.3. *There are $p(n)$ type I and type II partitions.*

Proof. Let

$$2n = (n+1)\, j_{n+1} + nj_n + \cdots + 2j_2 + j_1 \tag{1-1}$$

be a type I partition, i.e.,

$$n = j_{n+1} + \cdots + j_1. \tag{1-2}$$

Subtracting (1-2) from (1-1), we have

$$n = nj_{n+1} + (n-1)\, j_n + \cdots + j_2. \tag{1-3}$$

Therefore, $j_2, ..., j_{n+1}$ determine a partition of n. Different type I partitions (1-1) yield different partitions (1-3).

Conversely, given a partition (1-3) of n, define $j_1 = 2n - (n+1)\, j_{n+1} - \cdots - 2j_2$. Then $j_1 + \cdots + j_{n+1} = n$ and $j_1, ..., j_{n+1}$ determine a type I partition as in (1-1). Again, different partitions (1-3) yield different type I partitions (1-1). Hence there are $p(n)$ type I partitions.

There is a one-to-one connection between type I and type II partitions. Let the Ferrers graph of a type I partition be given by

$$\begin{array}{c|ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & & \\
 & \vdots & & & & \\
\bullet & \bullet & & & & \\
\bullet & & & & &
\end{array} \tag{1-4}$$

Since this represents a type I partition there are exactly n rows. We now remove the first column and place it below the second column to obtain

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & & & \\
\vdots & & & & \\
\bullet & & & & \\
\hline
\bullet & & & & \\
\bullet & & & & \\
\bullet & & & & \\
\vdots & & & & \\
\bullet & & & & \\
\bullet & & & &
\end{array} \tag{1-5}$$

This is clearly a type II partition. For distinct type I partitions (1-4) there are distinct type II partitions (1-5) and vice versa. Therefore, there are also $p(n)$ type II partitions. ▮

Remark 1.4. Let $\alpha = (\alpha_1, ..., \alpha_m, 2, ..., 2)$, $\alpha_i \geqslant 3$ for $1 \leqslant i \leqslant m$, be a type I partition and α^* be the associated type II partition as in the proof of the proposition. Then $\alpha^* = (\alpha_1 - 1, ..., \alpha_m - 1)$.

Let $\beta^1, ..., \beta^{p(n)}$ be the type I and $\alpha^1, ..., \alpha^{p(n)}$ be the type II partitions, arranged in decreasing lexicographical order. Let m_{ij} be the number of ways to refine β^j into α^i, ignoring order, $1 \leqslant i, j \leqslant p(n)$ (if α^i is not a refinement of β^j then $m_{ij} = 0$). The $(p(n) \times p(n))$ matrix with entries m_{ij} is denoted by M_n, e.g.

$$M_4 = \begin{array}{c|ccccc} & (5) & (4,2) & (3,3) & (3,2,2) & (2,2,2,2) \\ \hline (4) & 5 & 1 & 0 & 0 & 0 \\ (3) & 10 & 4 & 2 & 1 & 0 \\ (2,2) & 15 & 9 & 9 & 7 & 6 \\ (2) & 10 & 7 & 6 & 5 & 4 \\ (\,) & 1 & 1 & 1 & 1 & 1 \end{array} \tag{1-6}$$

The type I and type II partitions for $n=4$ can be found at the top and at the left of the matrix, respectively.

The main goal of this paper is to prove (see §3):

THEOREM 1.5.

$$\det M_n = 1, \qquad \textit{for all } n.$$

Surprisingly, the motivation for this theorem was not combinatorial or number-theoretic in nature. The theorem can be used to show that certain functions cannot satisfy algebraic differential equations. This will be the subject of another paper (see [R]).

2. Combinatorial Considerations

Let $\mathscr{S}$ be the set of all finite tuples of integers $\geqslant 2$ arranged in non-increasing order. The length of $\alpha = (\alpha_1, ..., \alpha_m) \in \mathscr{S}$ is defined by $|\alpha| = m$ and the weight by $w(\alpha) = \sum_{i=1}^{m} \alpha_i$. The empty tuple will be denoted by ( ) (its length and weight are 0). If α, $\beta \in \mathscr{S}$, then the product $\alpha\beta$ is a tuple of length $|\alpha| + |\beta|$ consisting of the integers of α and β arranged in non-increasing order. Powers of tuples can be defined accordingly. For

$$\alpha = (\underbrace{\alpha_1, ..., \alpha_1}_{j_1 \text{ times}}, ..., \underbrace{\alpha_l, ..., \alpha_l}_{j_l \text{ times}})$$

we also write $\alpha = (\alpha_1^{j_1}, ..., \alpha_l^{j_l})$, where $\alpha_i \neq \alpha_j$ if $i \neq j$. The factorial factor of α is defined to be

$$FF(\alpha) = \frac{1}{j_1! \cdots j_l!}.$$

The factorial factor of the empty tuple is set to be 1. In this notation $|\alpha| = \sum_{i=1}^{l} j_i$ and $w(\alpha) = \sum_{i=1}^{l} j_i \alpha_i$.

DEFINITION 2.1. We define two orders on $\mathscr{S}$:

(a) $\alpha=(\alpha_1, ..., \alpha_m)$ is said to be of higher l-order than $\beta=(\beta_1, ..., \beta_k)$ (written $\alpha >^l \beta$, l for lexicographical order), if $\alpha_1 > \beta_1$, or if $\exists i$ s.t. $\alpha_j=\beta_j$ for $a \leqslant j \leqslant i-1$ but $\alpha_i > \beta_i$, or if $|\alpha| > |\beta|$ and $\alpha_j=\beta_j$ for $1 \leqslant i \leqslant k$. $(\ ) <^l \alpha$ if $|\alpha| \geqslant 1$.

(b) α is said to be of higher dl-order than β (written $\alpha >^{dl} \beta$, dl for degree-lexicographical order), if $|\alpha| > |\beta|$, or if $|\alpha| = |\beta|$ and $\alpha >^l \beta$. $(\ ) <^{dl} \alpha$ if $|\alpha| \geqslant 1$.

DEFINITION 2.2. $\bar{\mu}=(\mu_1, ..., \mu_p)$, where the μ_i are tuples, is called a *decomposition* of α if $\mu_1 \cdots \mu_p = \alpha$. For

$$\bar{\mu}=(\underbrace{\mu_1, ..., \mu_1}_{j_1 \text{ times}}, ..., \underbrace{\mu_p, ..., \mu_p}_{j_p \text{ times}})$$

where $\mu_i \neq \mu_j$ if $i \neq j$, we also write $\bar{\mu}=(\mu_1^{j_1}, ..., \mu_p^{j_p})$. The factorial factor of $\bar{\mu}$ is defined by

$$FF(\bar{\mu}) = \frac{1}{j_1! \cdots j_p!}.$$

Furthermore the length of $\bar{\mu}$ is $|\bar{\mu}| = \sum_{i=1}^p j_i$.

Remark 2.3. We use the following convention: superscripts occurring in tuples do not mean exponents but rather the number of times the entry is repeated. Otherwise they mean exponents (powers). If $\mu_1=(4, 2)$ and $\mu_2=(3, 2)$, then $\mu_1^2\mu_2$ represents the tuple (4, 4, 3, 2, 2, 2), whereas (μ_1^2, μ_2) is the decomposition ((4, 2), (4, 2), (3, 2)).

If the tuples in a decomposition $\bar{\mu}=(\mu_1, ..., \mu_p)$ are arranged such that $w(\mu_i) \geqslant w(\mu_j)$ if $i > j$, then decompositions can be ordered in the following way (called λ-rank for reasons to be explained later):

DEFINITION 2.4. Let $\bar{\mu}=(\mu_1, ..., \mu_p)$ and $\bar{\eta}=(\eta_1, ..., \eta_q)$. $\bar{\mu}$ is said to be of higher λ-rank ($\bar{\mu} >^\lambda \bar{\eta}$) if $(w(\mu_1), ..., w(\mu_p)) >^{dl} (w(\eta_1), ..., w(\eta_q))$.

PROPOSITION 2.5. *Let* $\alpha=(\alpha_1^{j_1}, ..., \alpha_m^{j_m})$. $\bar{\mu}=((\alpha_1)^{j_1}, ..., (\alpha_m)^{j_m})$ *is the decomposition of* α *of highest* λ*-rank.*

Proof. If $\bar{\eta}$ is a decomposition of α, then $|\bar{\eta}| \leqslant |\alpha|$ and $|\bar{\eta}| = |\alpha|$ iff $\bar{\eta}=\bar{\mu}$. ∎

DEFINITION 2.6. (a) Let $n \geqslant 1$ be an integer and $\alpha = (\alpha_1, ... \alpha_m)$ s.t. $w(\alpha) \leqslant n$. Define

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1! \cdots \alpha_m! \, (n - \alpha_1 - \cdots - \alpha_m)!}.$$

If $w(\alpha) > n$ define $\binom{n}{\alpha} = 0$.

(b) Define

$$\lambda_\alpha^{(n)} = FF(\alpha) \binom{n}{\alpha}.$$

Remark 2.7. (a) The superscript is omitted if it is clear what n is.

(b) It follows from Theorem 13.2 in [A] that $\lambda_\alpha^{(n)}$ is the number of ways to refine (n) into α, ignoring order, if (n) and α are considered as partitions of n.

(c) $\lambda_\alpha^{(n)}$ does not depend on the number of 1's in the partition α. This is the major reason why all entries in tuples are greater than 1.

(d) $\lambda_\alpha^{(n)} = 0$ if $w(\alpha) > n$ and α is not a refinement of (n) in this case.

PROPOSITION 2.8.

$$\lambda_\alpha^{(n)} = \lambda_\alpha^{(w(\alpha))} \lambda_{(w(\alpha))}^{(n)}.$$

Proof. Let $\alpha = (\alpha_1, ..., \alpha_m)$.

$$\lambda_\alpha^{(n)} = FF(\alpha) \frac{n!}{\alpha_1! \cdots \alpha_m! \, (n - w(\alpha))!} = \frac{FF(\alpha) \, w(\alpha)!}{\alpha_1! \cdots \alpha_m!} \frac{n!}{w(\alpha)! \, (n - w(\alpha))!}$$

The proposition follows from the fact that

$$\frac{FF(\alpha) \, w(\alpha)!}{\alpha_1! \cdots \alpha_m!} = \lambda_\alpha^{(w(\alpha))} \quad \text{and} \quad \frac{n!}{w(\alpha)! \, (n - w(\alpha))!} = \lambda_{(w(\alpha))}^{(n)}. \quad \blacksquare$$

DEFINITION 2.9. Let α be a tuple and $\beta = (\beta_1, ..., \beta_k)$. Define

$$\Lambda_\alpha^\beta = \sum_{\bar{\mu} = (\mu_1, ..., \mu_p)} \sum_{l_1 \neq \cdots \neq l_p} FF(\bar{\mu}) \, \lambda_{\mu_1}^{(\beta_{l_1})} \cdots \lambda_{\mu_p}^{(\beta_{l_p})}, \tag{2-1}$$

where the first sum ranges over all decompositions of α and the second sum over all $l_1, ..., l_p$, $1 \leqslant l_i \leqslant k$, s.t. $l_i \neq l_j$ if $i \neq j$. Define $\Lambda_{()}^\beta = 1$ for all tuples β.

Remark 2.10.

(a) We usually omit β and write (2-1) in the form

$$\Lambda_\alpha = \sum_{\bar{\mu}} FF(\bar{\mu})\, \lambda_{\mu_1} \cdots \lambda_{\mu_p}, \tag{2-2}$$

where $\bar{\mu}$ is a decomposition of α and $\bar{\mu} = (\mu_1, ..., \mu_p)$.

(b) The λ-rank of a term $\lambda_{\mu_1} \cdots \lambda_{\mu_p}$ is defined to be the λ-rank of $\bar{\mu}$.

PROPOSITION 2.11. *Λ_α^β is the number of ways to refine β into α ignoring order.*

Proof. Let $\bar{\mu} = (\mu_1, ..., \mu_p)$ be a composition of α. There are $\lambda_{\mu_1}^{(\beta_1)} \lambda_{\mu_2}^{(\beta_2)} \cdots \lambda_{\mu_p}^{(\beta_p)}$ possibilities to refine (β_1) into μ_1, (β_2) into μ_2 and so on. Some of the tuples $\mu_1, ..., \mu_p$ might be equal. Therefore, we have to multiply by $FF(\bar{\mu})$ to ignore the order in which the tuples are arranged. Hence

$$\sum_{l_1 \neq \cdots \neq l_p} FF(\bar{\mu})\, \lambda_{\mu_1}^{(\beta_{l_1})} \cdots \lambda_{\mu_1}^{(\beta_{l_p})}, \tag{2-3}$$

gives the number of ways to refine some (β_{l_1}) into μ_1, another (β_{l_2}) into μ_2 etc, ignoring order. Summing (2-3) over all decompositions of α gives all possibilities to refine β into α, ignoring order. ∎

Remark 2.12. If α is not a refinement of β, then $\Lambda_\alpha^\beta = 0$. Especially if $\alpha >^l \beta$ then $\Lambda_\alpha^\beta = 0$.

3. PROOF OF THE MAIN THEOREM

Throughout this section, n denotes a fixed positive integer. Recall that $\beta^1, ..., \beta^{p(n)}$ and $\alpha^1, ..., \alpha^{p(n)}$ are type I and type II partitions, arranged in decreasing l-order. Let M_n be the matrix whose entry in the i-th row and j-th column is the number of ways to refine β^j into α^i. We want to show that $\det M_n = 1$. The proof will be done in several steps.

LEMMA 3.1. *Let $\alpha = (\alpha_1, ..., \alpha_m)$. Then*

$$\Lambda_\alpha = FF(\alpha)\, \lambda_{(\alpha_1)} \cdots \lambda_{(\alpha_m)} + \textit{terms of lower } \lambda\textit{-rank}.$$

Proof. Follows from Proposition 2.5. ∎

Remark 3.2. We consider sums of the form

$$\sum_{\bar{\mu}} c_{\bar{\mu}} \lambda_{\mu_1} \cdots \lambda_{\mu_p}, \; c_{\bar{\mu}} \in \mathbb{R}. \tag{3-1}$$

(3-1) is an abbreviation for a double sum in the sense of (2-1) and (2-2). If there are several terms of highest λ-rank in (3-1), they can be rewritten into one single term by Proposition 2.8, e.g., $\lambda_{(3,3)}\lambda_{(2)}$ and $\lambda_{(4,2)}\lambda_{(2)}$ are of the same λ-rank, but $\lambda_{(3,3)}\lambda_{(2)}+\lambda_{(4,2)}\lambda_{(2)}=(\lambda^{(6)}_{(3,3)}+\lambda^{(6)}_{(4,2)})\ \lambda_{(6)}\lambda_{(2)}=25\lambda_{(6)}\lambda_{(2)}$.

In general, a term $A=c_{\bar{\mu}}\lambda_{\mu_1}\cdots\lambda_{\mu_p}$ becomes a term $B=dc_{\bar{\mu}}\,\lambda_{(w(\mu_1))}\cdots\lambda_{(w(\mu_p))}$, where $d\in\mathbb{Z}$. Notice that A and B are of the same λ-rank. If $\alpha=(\alpha_1,...,\alpha_m)$ is a tuple, $\bar{\alpha}=((\alpha_1),...,(\alpha_m))$ denotes the decomposition of α of highest λ-rank. In this way, we may assume that if $c_{\bar{\alpha}}\lambda_{(\alpha_1)}\cdots\lambda_{(\alpha_m)}$ is a highest term in (3-1), then it is the only term of highest λ-rank and the (α_i) are tuples of length 1.

Lemma 3.3. *Given a sum* (3-1). *Let α and $\bar{\alpha}$ be as in the previous remark. Then*

$$\sum_{\bar{\mu}} c_{\bar{\mu}}\lambda_{\mu_1}\cdots\lambda_{\mu_p}=\frac{1}{FF(\alpha)}\,c_{\bar{\alpha}}\Lambda_{\alpha}+\sum_{\gamma} d_{\gamma}\Lambda_{\gamma},$$

where $d_{\gamma}\in\mathbb{R}$ and $\gamma<^{dl}\alpha$.

Proof. Induction on the λ-rank of α. $c\lambda_{(2)}$ is of lowest λ-rank of all possible sums (3-1) and $c\lambda_{(2)}=c\Lambda_{(2)}$.

For the induction step consider

$$\sum_{\bar{\mu}} c_{\bar{\mu}}\lambda_{\mu_1}\cdots\lambda_{\mu_p}=c_{\bar{\alpha}}\lambda_{(\alpha_1)}\cdots\lambda_{(\alpha_m)})+\text{terms of lower }\lambda\text{-rank}. \tag{3-2}$$

By Lemma 3.1

$$c_{\bar{\alpha}}\lambda_{(\alpha_1)}\cdots\lambda_{(\alpha_m)}=\frac{1}{FF(\alpha)}\,c_{\bar{\alpha}}\Lambda_{\alpha}+\text{terms of lower }\lambda\text{-rank}.$$

Therefore by (3-2)

$$\sum_{\bar{\mu}} c_{\bar{\mu}}\lambda_{\mu_1}\cdots\lambda_{\mu_p}=\frac{1}{FF(\alpha)}\,c_{\bar{\alpha}}\Lambda_{\alpha}+\text{terms of lower }\lambda\text{-rank}.$$

By induction and Remark 3.2, the "terms of lower λ-rank" can be written in the form $\sum_{\gamma} d_{\gamma}\Lambda_{\gamma}$, where all $\gamma<^{dl}\alpha$ and the result follows. ∎

Lemma 3.4. *Let* $\nu=(\nu_1,...,\nu_l)$, $\nu_i\geqslant 3$ *and* $\eta^{(i)}=(\nu_1,...,\nu_i-1,...,\nu_l)$ (*notice that* $\eta^{(i)}<^{dl}\nu$). *Then*

(a)
$$\binom{h}{\nu}=\binom{h-1}{\nu}+\sum_{i=1}^{l}\binom{h-1}{\eta^{(i)}}.$$

(b)
$$\lambda_\nu^{(h)}=\lambda_\nu^{(h-1)}+\sum_{i=1}^{l} d_i\lambda_{\eta^{(i)}}^{(h-1)}, \tag{3-3}$$

where $d_i\in\mathbb{Q}$.

Proof. (a)

$$\begin{aligned}\binom{h-1}{\nu}+\sum_{i=1}^{l}\binom{h-1}{\eta^{(i)}}&=\frac{(h-1)!}{\nu_1!\cdots\nu_l!\,(h-1-\nu_1-\cdots-\nu_l)!}\\&\quad+\sum_{i=1}^{l}\frac{(h-1)!}{\nu_1!\cdots(\nu_i-1)!\cdots\nu_l!\,(h-\nu_1-\cdots-\nu_l)!}\\&=\frac{(h-1)!\,(h-\nu_1-\cdots-\nu_l)}{\nu_1!\cdots\nu_l!\,(h-\nu_1-\cdots-\nu_l)!}\\&\quad+\sum_{i=1}^{l}\frac{(h-1)!\,\nu_i}{\nu_1!\cdots\nu_l!\,(h-\nu_1-\cdots-\nu_l)!}\\&=\frac{(h-1)!\,h}{\nu_1!\cdots\nu_l!\,(h-\nu_1-\cdots-\nu_l)!}=\binom{h}{\nu}\end{aligned}$$

(b)
$$\begin{aligned}\lambda_\nu^{(h)}&=FF(\nu)\binom{h}{\nu}=FF(\nu)\binom{h-1}{\nu}+\sum_{i=1}^{l}FF(\nu)\binom{h-1}{\eta^{(i)}}\\&=\lambda_\nu^{(h-1)}+\sum_{i=1}^{l}\frac{FF(\nu)}{FF(\eta^{(i)})}\lambda_{\eta^{(i)}}^{(h-1)}.\quad\blacksquare\end{aligned}$$

LEMMA 3.5. *Let* $\beta=(\beta_1,...,\beta_k)$ *be a type I partition* (*i.e. a partition of* $2n$ *into exactly* n *parts*). *Then*

$$\sum_{i=1}^{k}(\beta_i-1)=n.$$

Proof. Let $\beta_{k+1}=\cdots=\beta_n=1$. Then $\sum_{i=1}^{n}\beta_i=2n$.

$$\sum_{i=1}^{k}(\beta_i-1)=\sum_{i=1}^{n}(\beta_i-1)=2n-n=n.\quad\blacksquare$$

Remark 3.6. It is clear that a tuple α represents a type II partition iff $w(\alpha)\leqslant n$.

LEMMA 3.7. *Let $\beta = (\beta_1, ..., \beta_k)$ be a type I partition, β^* the type II partition associated with β and $\alpha = (\alpha_1, ..., \alpha_m)$ be a type II partition. Then*

$$\Lambda_\alpha^\beta = \Lambda_\alpha^{\beta^*} + \sum_\gamma c_\gamma \Lambda_\gamma^{\beta^*}, \tag{3-4}$$

where $c_\gamma \in \mathbb{R}$ (actually $\in \mathbb{Z}$ as wil be seen later), and the γ's range over type II partitions s.t. $\gamma <^{dl} \alpha$; some $\gamma = (\ )$ is possible.

Proof. *Case* I: $\alpha_i \geqslant 3$ for $1 \leqslant i \leqslant m$. By Lemma 3.1

$$\begin{aligned} \Lambda_\alpha &= \sum_{\bar{\mu}} FF(\bar{\mu})\, \lambda_{\mu_1} \cdots \lambda_{\mu_p} \\ &= FF(\alpha)\, \lambda_{(\alpha_1)} \cdots \lambda_{(\alpha_m)} + \text{terms of lower } \lambda\text{-rank}. \end{aligned} \tag{3-5}$$

The first summand in (3-5) is the short form for

$$FF(\alpha)\, \lambda_{(\alpha_1)} \cdots \lambda_{(\alpha_m)} = \sum_{l_1 \neq \cdots \neq l_m} FF(\alpha)\, \lambda_{(\alpha_1)}^{(\beta_{l_1})} \cdots \lambda_{(\alpha_m)}^{(\beta_{l_m})}. \tag{3-6}$$

By Lemma 3.4 (applied for $v = (\alpha_i)$)

$$\lambda_{(\alpha_i)}^{(\beta_{l_i})} = \lambda_{(\alpha_i)}^{(\beta_{l_i} - 1)} + d\lambda_{(\alpha_i - 1)}^{(\beta_{l_i} - 1)}, \tag{3-7}$$

where $d \in \mathbb{Q}$. Substituting (3-7) into (3-6) for $1 \leqslant i \leqslant m$ we get

$$\begin{aligned} \sum_{l_1 \neq \cdots \neq l_m} FF(\alpha)\, \lambda_{(\alpha_1)}^{(\beta_{l_1})} \cdots \lambda_{(\alpha_m)}^{(\beta_{l_m})} &= \sum_{l_1 \neq \cdots \neq l_m} FF(\alpha)\, \lambda_{(\alpha_1)}^{(\beta_{l_1} - 1)} \cdots \lambda_{(\alpha_m)}^{(\beta_{l_m} - 1)} \\ &\quad + \text{terms of lower } \lambda\text{-rank}, \end{aligned} \tag{3-8}$$

where all superscripts of the λ's in the "terms of lower λ-rank" are of the form $\beta_{l_i} - 1$. Similarly, the "terms of lower λ-rank" in (3-5) can be rewritten by Lemma 3.4 such that the resulting terms are of lower λ-rank than α and the superscripts of the λ's are of the form $\beta_{l_i} - 1$.

Notice that if $\beta_i = 2$ in β, then β_i is dropped in β^*, but then $\lambda_v^{(\beta_i - 1)} = \lambda_v^{(1)} = 0$ for any nonempty tuple v. Therefore, the sum on the right hand side of (3-8) can be considered as ranging over β^*.

It now follows from (3-5) and Lemma 3.3 applied to the right hand side of (3-8) that

$$\Lambda_\alpha^\beta = \Lambda_\alpha^{\beta^*} + \sum_\gamma c_\gamma \Lambda_\gamma^{\beta^*},$$

where $d_\gamma \in \mathbb{R}$ and $\gamma <^{dl} \alpha$. It is clear that $w(\gamma) \leqslant w(\alpha)$ and hence the γ's are type II partitions by Remark 3.6.

Case II: some entries of α are 2. Then some expressions in (3-3) are not defined. Let $v=(v_1, ..., v_l, 2)$, $v_l \geqslant 2$. Then $\lambda^{(h-1)}_{(v_1, ..., v_l, 1)}$ occurs in (3-3). This expression stands for

$$\begin{aligned}\lambda^{(h-1)}_{(v_1, ..., v_l, 1)} &= FF((v_1, ..., v_l)) \frac{(h-1)!}{v_1! \cdots v_l! \, (h-2-v_1-\cdots-v_l)!} \\ &= \frac{FF((v_1, ..., v_l))(h-1)! \, (v_l+1)}{v_1! \cdots v_{l-1}! \, (v_l+1)! \, (h-2-v_1-\cdots-v_l)!} \\ &= d\lambda^{(h-1)}_{(v_1, ..., v_{l-1}, v_l+1)},\end{aligned}$$

where $d \in \mathbb{Q}$. We may rearrange the entries of $(v_1, ..., v_{l-1}, v_l+1)$ in non-increasing order, since the definition of λ does not depend on the order the entries of partitions are arranged. In this way, all $\lambda^{(h-1)}_{(v_1, ..., v_l, 1)}$ can be defined and again there is a relation (3-3).

If $(v)=2$ this procedure cannot be applied. Let $\alpha=(\alpha_1, ..., \alpha_m, 2, ..., 2)$. We have

$$\lambda^{(h)}_{(2)} = \binom{h}{2} = \binom{h-1}{2} + h - 1 = \lambda^{(h-1)}_{(2)} + h - 1. \tag{3-9}$$

Again, we consider (3-5), plug in (3-3) and (3-9) and get a relation (3-8). It is possible that there are terms of the form

$$\sum_{l_1 \neq l_p} c_{\bar{\mu}} \lambda^{(\beta_{l_1}-1)}_{\mu_1} \cdots \lambda^{(\beta_{l_r}-1)}_{\mu_r} (\beta_{l_{r+1}}-1) \cdots (\beta_{l_p}-1) \tag{3-10}$$

in the "terms of lower λ-rank" in (3-8). If there are $p-r$ 2's in α, then there are at most $p-r$ factors of the form $(\beta_{l_i}-1)$ in any term (3-10).

We now show that (3-10) can be written in the form

$$\sum_{\bar{v}} \sum_{l_1 \neq \cdots \neq l_q} c_{\bar{v}} \lambda^{(\beta_{l_1}-1)}_{v_1} \cdots \lambda^{(\beta_{l_q}-1)}_{v_q}, \tag{3-11}$$

where the $\bar{v}$ are of lower λ-rank than $\bar{\alpha}$. (3-10) equals

$$\sum_{l_1 \neq \cdots \neq l_{p-1}} \underbrace{c_{\bar{\mu}} \lambda^{(\beta_{l_1}-1)}_{\mu_1} \cdots \lambda^{(\beta_{l_r}-1)}_{\mu_r} (\beta_{l_{r+1}}-1) \cdots (\beta_{l_{p-1}}-1)}_{A} \sum_{l_p \neq l_1, ..., l_{p-1}} (\beta_{l_p}-1).$$

By Lemma 3.5

$$\sum_{l_p \neq l_1, ..., l_{p-1}} (\beta_{l_p}-1) = (n-(\beta_{l_1}-1)-\cdots-(\beta_{l_{p-1}}-1)) = n+p-1-\sum_{j=1}^{p-1} \beta_{l_j}.$$

Hence (3-10) equals

$$(n+p-1) \sum_{l_1 \neq \cdots \neq l_{p-1}} A - \sum_{j-1}^{r} \sum_{l_1 \neq \cdots \neq l_{p-1}} A\beta_{l_j} - \sum_{j=r+1}^{p-1} \sum_{l_1 \neq \cdots \neq l_{p-1}} A\beta_{l_j}. \tag{3-12}$$

Now we use an inductive argument. Since A involves fewer $(\beta_{l_j}-1)$ factors than (3-10), the first sum in (3-12) can be written in the required form by induction.

Consider $j=1$ in the second sum of (3-14). $\beta_{l_1}\lambda_{\mu_1}^{(\beta_{l_1}-1)}$ is a factor of $A\beta_{l_1}$. Let $\mu_1=(\eta_1, ..., \eta_s)$. Then

$$\begin{aligned}
\beta_{l_1}\lambda_{\mu_1}^{(\beta_{l_1}-1)} &= FF(\mu_1)\, \beta_{l_1} \frac{(\beta_{l_1}-1)!}{\eta_1! \cdots \eta_s!\, (\beta_{l_1}-1-\eta_1- \cdots \eta_s)!} \\
&= \frac{FF(\mu_1)\, \beta_{l_1}!\, (\eta_1+1)}{(\eta_1+1)!\, \eta_1! \cdots \eta_s!\, (\beta_{l_1}-1-\eta_1- \cdots \eta_s)!} \\
&= d\lambda_{(\eta_1+1, \eta_2, ..., \eta_s)}^{(\beta_{l_1})},
\end{aligned}$$

where $d \in \mathbb{Q}$. Apply Lemma 3.4 to $\lambda_{(\eta_1+1, \eta_2, ..., \eta_s)}^{(\beta_{l_1})}$. This way $\beta_{l_1}A$ can be written as a sum of terms of the form (3-10) with fewer $(\beta_{l_j}-1)$ factors. Let $\bar{\mu}_1=(\eta_1+1, \eta_2, ..., \eta_s)$. Notice that $(\tilde{\mu}_1, \mu_2, ..., \mu_r, (2), ..., (2))$, $(p-r-1)$ of the 2's, has shorter length than $\bar{\alpha}$ and is hence of lower *dl*-order than $\bar{\alpha}$. Therefore, the second sum in (3-12) can be written in the required form by induction.

In the third sum of (3-14) notice that, e.g.

$$\beta_{l_{p-1}}(\beta_{l_{p-1}}-1) = 2\binom{\beta_{l_{p-1}}}{2} = 2\binom{\beta_{l_{p-1}}-1}{2} - 2(\beta_{l_{p-1}}-1).$$

Hence

$$\begin{aligned}
A\beta_{lp-1} = {} & 2c_{\bar{\mu}}\lambda_{\mu_1}^{(\beta_{l_1}-1)} \cdots \lambda_{\mu_r}^{(\beta_{l_r}-1)} \lambda_{(2)}^{(\beta_{l_{p-1}}-1)} (\beta_{l_{r+1}}-1) \cdots (\beta_{l_{p-2}}-1) \\
& - 2c_{\bar{\mu}}\lambda_{\mu_1}^{(\beta_{l_1}-1)} \cdots \lambda_{\mu_r}^{(\beta_{l_r}-1)} (\beta_{l_{r+1}}-1) \cdots (\beta_{l_{p-1}}-1).
\end{aligned} \tag{3-13}$$

Notice that the first summand of (3-13) has two fewer factors $(\beta_{l_j}-1)$ than (3-10) but an extra $\lambda_{(2)}^{(\beta_{l_{p-1}}-1)}$. This means the decomposition of highest rank that can occur in (3-11) is $(\alpha_1, ..., \alpha_m, 2, ..., 2)$, where there are at most $[(p-r)/2]$ of the 2's. Since this tuple has shorter length than α, it is of lower *dl*-order than α. Hence the third sum in (3-12) can be written in the form (3-11) by induction.

Thus, in the case that some entries of α are 2, there is also a relation (3-8) and all terms are well defined. The result now follows by Case I.

Notice that if (3-10) is just the sum $\sum_{l_1}(\beta_{l_1}-1)$ then $\sum_{l_1}(\beta_{l_1}-1)= n=n\Lambda_{()}$, so $\Lambda_{()}$ may actually occur. ∎

Proof of Theorem 1.5. *Claim:*

$$\Lambda_{\alpha_i}^{\beta_j^*}=\Lambda_{\alpha_i}^{\beta_j}+\sum_{\mu} c_\mu \Lambda_{\mu}^{\beta_j}, \tag{3-14}$$

where $c_\mu \in \mathbb{R}$ and $\mu <^{dl} \alpha_i$.

Induction on the *dl*-order of α_i. If $\alpha_i=(\ )$ then $1=\Lambda_{()}^{\beta_j^*}=\Lambda_{()}^{\beta_j}$.

Now let $\alpha \neq (\ )$. By Lemma 3.7 we have

$$\Lambda_{\alpha_i}^{\beta_j^*}=\Lambda_{\alpha_i}^{\beta_j}-\sum_{\mu} c_\mu \Lambda_{\mu}^{\beta_j^*},$$

where $c_\mu \in \mathbb{R}$ and $\mu <^{dl} \alpha_i$. By the induction hypothesis

$$\Lambda_{\mu}^{\beta_j^*}=\Lambda_{\mu}^{\beta_j}+\sum_{\nu} d_\nu \Lambda_{\nu}^{\beta_j},$$

where $\nu <^{dl} < \mu^{dl}\, \alpha_i$. This establishes the claim.

The claim means that if we add to the i'th row multiples of rows corresponding to α's of lower *dl*-order, we obtain a row with entries

$$\Lambda_{\alpha_i}^{\beta_j^*}=\Lambda_{\alpha_i}^{\alpha_j}.$$

But if $j>i$ then $\alpha_j <^{l} \alpha_i$ and α_i is not a refinement of α_j. Hence $\Lambda_{\alpha_i}^{\alpha_j}=0$. Obviously, $\Lambda_{\alpha_i}^{\alpha_i}=1$, for $1 \leqslant i \leqslant p(n)$. Thus the matrix with entries $\Lambda_{\alpha_i}^{\alpha_j}$ is a lower triangular matrix with 1's in its main diagonal. The new matrix was obtained from the original one by only adding rows to other rows. Therefore, $\det M_n=1$. ∎

Remark 3.8. We now show that the c_μ in (3-4), and hence in (3-14), can be taken to be integers.

Proof. This follows form the following fact. WLOG we assume that the rows of M_n are in decreasing *dl*-order. Let M_n be a $(p \times p)$ matrix, its entries be denoted by m_{ij} and its rows simply by m_i. By the proof of Theorem 1.5 there are relations

$$\begin{aligned} n_p &= m_p \\ n_{p-1} &= m_{p-1}+d_p^{(p-1)} n_p \\ &\vdots \\ n_1 &= m_1+\sum_{k=2}^{p} d_k^{(1)} n_k, \end{aligned} \tag{3-15}$$

where $d_k^{(j)} \in \mathbb{R}$, so that the matrix with rows $n_1, ..., n_p$ is a lower triangular matrix with integer entries and 1's in its main diagonal. Let this matrix be N with entries n_{ij}.

Induction on the number of the row. $n_p = m_p$: nothing to show. After having changed rows $m_{i+1}, ..., m_p$ to rows $n_{i+1}, ..., n_p$ according to (3-15), we get a matrix of the form

$$\begin{pmatrix} & & * & & & \\ m_{i1} & \cdots & m_{i(i+1)} & & \cdots & m_{ip} \\ & & 1 & 0 & \cdots & 0 \\ & * & & \ddots & & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

Now suppose $d_p^{(i)}, ..., d_{j+1}^{(i)} \in \mathbb{Z}$, but $d_j^{(i)} \notin \mathbb{Z}$ for some $j \geqslant i+1$. Then $n_{rj} \in \mathbb{Z}$ for $j+1 \leqslant r \leqslant p$, $n_{jj} = 1$ and $n_{rj} = 0$ for $i+1 \leqslant r \leqslant j-1$ by induction. Hence

$$n_{ij} = m_{ij} + \sum_{k=i+1}^{p} d_k^{(i)} n_{kj} = \text{integer} + d_j^{(i)} \neq 0.$$

But $n_{ij} = 0$ since $j \geqslant i+1$ nd the matrix N is triangular. ∎

EXAMPLE 3.10. The following row operations can be applied to the matrix M_4 at the end of Section 1.

$$n_5 = m_5$$

$$n_4 = m_4 - 4m_5$$

$$n_3 = m_3 - m_4 - 2m_5$$

$$m_2 = m_2 - m_4 + 4m_5$$

$$m_1 = m_1 - m_2 + m_4 - 4m_5$$

to get the following matrix

	(4)	(3)	(2, 2)	(2)	( )
(4)	1	0	0	0	0
(3)	4	1	0	0	0
(2, 2)	3	0	1	0	0
(2)	6	3	2	1	0
( )	1	1	1	1	1

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